Gauge Invariance in Field Theory and Statistical Physics in Operator Formalism

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Abstract

We obtain the Ward identities and the gauge-dependence of Green's functions in non-Abelian gauge theories by using only the canonical commutation relations and the equations of motion for the Heisenberg operators. The consideration is applicable to theories both with and without spontaneous symmetry breaking. We present a definition of a generalized statistical average which ensures that the Fourier images of temperature Green's functions of the Fermionic fields have only even-valued frequencies. This makes it possible to set up a procedure of gauge-invariant statistical averaging in terms of the Hamiltonian and the field operators.

1 Introduction

The study of the effects of spontaneously-broken gauge theories in statistical physics [1] has raised the problem of finding a proof of the gauge-invariance of physical results in gauge statistical physics. Formally, such a proof can be carried out by analogy with quantum field theory, by using a representation of statistical averaging with the help of functional integration, which has been presented in [2]. However, there exist some specific calculations of physical effects [3] that have an appearance of being gauge-dependent. This circumstance may cast a shadow on the applicability of the functional approach to statistical physics. In particular, one may raise the question as to the validity of a non-local change of variables in the functional integral for the partition function, which has to be made in the course of the usual proof of gauge-invariance, as well as in the process of deriving the Ward identities.

It appears useful, therefore, to deduce the Ward identities and the gauge-invariance of physical results in the framework of the operator formalism of quantum field theory, i.e., by using only the Heisenberg equations of motion and canonical commutation relations. This has been done in the present article for both field theory (Sections 3, 4) and statistical physics (Section 5).

It is essential that the construction of a theory requires to sum up a gauge-invariant Lagrangian, L_0 , not only with a gauge-fixing term, but also with an additional Lagrangian, L_C , responsible for an interaction of fictitious particles with the gauge field. For those gauges that are usually applied in quantum electrodynamics, the fictitious particle is free, so that the theory can be set up without the term L_C . In the case of non-Abelian gauge theories, it is well-known that there arise some additional diagrams (with respect to the Feynman diagrams) that effectively describe an interaction of the gauge field with the fictitious particles. We choose the Lagrangian L_C in such a way that, on the one hand, it implies the necessary additional diagrams, and, on the other hand, it admits a canonical quantization.

The deduction of the Ward identities and the proof of gauge-invariance that are based on the equations of motion for the Heisenberg fields can also be useful for other purposes, such as a description of gauge theories in the framework of Zimmermann's normal product [4].

The fictitious particles described by the Lagrangian L_C are scalars, but, at the same time, they are Fermions. Thus, the usual definition of statistical average leads to such a Fourier-image of the temperature Green function of these particles that contains only odd-valued frequencies. However, as will be shown in Section 5, the gauge-invariance of the partition function demands that the Green function of fictitious particles should contain only even-valued frequencies (whereas the Green function of Bose particles should contain odd-valued frequencies). This allows one to pose statistics in terms of an operator formalism by using the Hamiltonian and the field operators. Physical quantities, and, in particular, the partition function,

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prove to be gauge-invariant. In the case of gauges that eliminate the fictitious particles, the generalized definition of the partition function becomes identical with the usual definition.

The present consideration has been made for a sufficiently large class of gauge conditions and is applicable to theories both with and without spontaneous symmetry breaking.

2 General formulas

In this section, we present some general formulas that we use in the following sections.

We examine a Lagrangian of a renormalizable theory of a gauge-invariant interaction of spinless and Fermionic fields with a gauge field of a general kind:

$$L = L_0 + \frac{1}{2} t^a \alpha_{ab} t^b + L_C \,, \tag{2.1}$$

$$L_{0} = -\frac{1}{4}G_{\mu\nu}^{a}G^{a,\mu\nu} + \frac{1}{2}\left[\left(\partial_{\mu} - ig\Gamma^{a}A_{\mu}^{a}\right)\varphi\right]^{2} + \overline{\psi}\gamma^{\mu}\left(i\partial_{\mu} + g\tau^{a}A_{\mu}^{a}\right)\psi - V\left(\psi,\varphi\right),$$

$$(2.2)$$

where φ_i is a multiplet of Hermitian spinless fields; ψ_i is a multiplet of Fermionic fields; $V(\psi,\varphi)$ is an arbitrary Lagrangian of interaction of the fields ψ , φ that obeys the requirement of gauge-invariance; $G^a_{\mu\nu}$ is the strength tensor of a Yang–Mills field A^a_{μ} :

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \,, \tag{2.3}$$

while f^{abc} are structure constants of an invariance group G; the matrices Γ^a and τ^a are the generators of the transformation group for the fields φ_i and ψ_i , with the commutation relations

$$\left[\Gamma^a, \Gamma^b\right] = i f^{abc} \Gamma^c \,, \tag{2.4}$$

and τ^a obey similar relations. The matrices Γ^a are antisymmetric and have purely imaginary values.

The last two terms in (2.1) describe subsidiary conditions that are necessary for the possibility of a canonical description of the theory (as well as for the existence of a perturbation theory). Let us choose the functions t^a in the form

$$t^a = \varkappa^{\mu\nu} \partial_\mu A^a_\nu + \varkappa^a_i \varphi_i \,, \tag{2.5}$$

where $\varkappa^{\mu\nu} = \varkappa^{\nu\mu}$ is a matrix that has numerical elements; the matrices \varkappa^a_i will be described below.

The Lagrangian L_C describes an interaction of fictitious particles (Fermionic scalars), C^a , C^{+a} , with the fields A^a_{μ} , φ_i and implies the well-known additional diagrams in quantum theory [5], namely,

$$L_C = -\partial_{\mu}C^{+a}\varkappa^{\mu\nu}\nabla^{ab}_{\nu}C^b + igC^{+a}\varkappa^a_i\Gamma^b_{ij}\varphi_jC^b,$$

$$\nabla^{ab}_{\nu} = \partial_{\nu}\delta^{ab} + gf^{acb}A^c_{\nu}.$$
(2.6)

The matrix \varkappa_i^a is chosen as follows: let the vacuum mean value of the field φ_i be non-vanishing,

$$\langle 0|\varphi_i|0\rangle = \xi_i. \tag{2.7}$$

The set of generators Γ_{ij}^a can always be split in two groups, $\Gamma^a = (\Gamma^m, \Gamma^l)$, such that

$$\Gamma^l \xi = 0, \ \Gamma^m \xi = i \xi^m \neq 0, \tag{2.8}$$

where the vectors ξ^m are orthogonal. Then \varkappa_i^a must obey the conditions

$$\varkappa_i^a \xi_i = 0. (2.9)$$

Besides, either (for a fixed a)

$$\varkappa_i^a \equiv 0 \,, \tag{2.10}$$

or

$$\varkappa_i^a \xi_i^m \neq 0, \tag{2.11}$$

with a certain m. In other respects, the matrix \varkappa_i^a is arbitrary. We do not discuss any restrictions that may be imposed on \varkappa_i^a by conditions (2.9)–(2.11). Notice that conditions (2.9) and (2.11) are necessary only

for the possibility of passing to unitary gauges of the kind $\varkappa_i^a \varphi_i = 0$. By themselves, they are unnecessary to provide the possibility of a canonical description.

In principle, one can choose a subsidiary condition of a yet more general kind, being compatible with renormalizability:

$$t \sim \partial A + \partial \varphi + A + \varphi + A^2 + A\varphi + \varphi^2$$
.

This, however, will only imply some obvious modifications of the reasonings to be presented below.

Since we examine a theory which admits a possibility of spontaneous symmetry breaking, let us now introduce some spinless fields, whose vacuum mean value is zero:

$$\varphi_i = \xi_i + \sigma_i, \ \langle 0|\sigma_i|0\rangle = 0.$$
 (2.12)

The Lagrangian L_0 is invariant with respect to gauge transformations whose infinitesimal form is given by

$$A^{a}_{\mu} \to A'^{a}_{\mu} = A^{a}_{\mu} + \nabla^{ab}_{\mu} \Lambda^{b} , \ \nabla^{ab}_{\mu} = \partial_{\mu} \delta^{ab} + g f^{acb} A^{c}_{\mu} ,$$

$$\psi \to \psi' = \psi + ig \Lambda^{a} \tau^{a} \psi , \ \varphi \to \varphi' = \varphi + ig \Lambda^{a} \Gamma^{a} \varphi ,$$

$$(2.13)$$

$$\psi \to \psi' = \psi + ig\Lambda^a \tau^a \psi, \ \varphi \to \varphi' = \varphi + ig\Lambda^a \Gamma^a \varphi,$$
 (2.14)

where Λ^a are infinitesimal parameters of the gauge transformations; they depend on the coordinates. For the field σ , transformation (2.14) reads as follows:

$$\sigma \to \sigma' = \sigma + ig\Lambda^a \Gamma^a \xi + ig\Lambda^a \Gamma^a \sigma, \qquad (2.15)$$

i.e., σ transforms in a non-homogenous manner. The invariance of L_0 with respect to (2.13)–(2.15) implies the identities

$$-\nabla_{\mu}^{ab}\frac{\delta L_{0}}{\delta A_{\mu}^{b}} + ig\frac{\delta L_{0}}{\delta \sigma_{i}}\Gamma_{ij}^{a}\varphi_{j} - ig\frac{\delta L_{0}}{\delta \psi_{i}}\tau_{ij}^{a}\psi_{j} - ig\overline{\psi}_{i}\tau_{ij}^{a}\frac{\delta L_{0}}{\delta \overline{\psi}_{j}} \equiv 0.$$
 (2.16)

Let us now proceed to the canonical quantization of the theory. The momenta of the fields A^a_μ , φ and ψ are defined as usual:

$$\pi_A: \pi^{a,0} = \alpha_{ab} \varkappa^{00} t^b, \ \pi^{a,k} = -G^{a,0k} + \alpha_{ab} \varkappa^{0k} t^b,$$
 (2.17)

$$\Pi_{\sigma}: \Pi_{i} = \dot{\sigma}_{i} - ig\Gamma_{ik}^{a} A_{0}^{a} \varphi_{k}, \qquad (2.18)$$

$$\Pi_{\psi}: \overline{\Pi}_{i} = i\psi_{i}^{+}, \qquad (2.19)$$

Using these relations, one can obtain the derivatives of the fields with respect to time:

$$t^{a} = \frac{1}{200} \alpha^{ab} \pi^{b,0} , \ \alpha^{ab} \alpha_{bc} = \delta^{a}_{c} , \tag{2.20}$$

$$G^{a,0k} = -\pi^{a,k} + \frac{1}{\varkappa^{00}} \varkappa^{0k} \pi^{a,0} ,$$

$$\dot{A}^{a,k} = \frac{\varkappa^{0k}}{\varkappa^{00}} \pi^{a,0} - \pi^{a,k} + \nabla^{ab,k} A^{b,0} , \qquad (2.21)$$

$$\dot{A}_0^a = \frac{1}{\varkappa^{00}} \left(\frac{1}{\varkappa^{00}} \alpha^{ab} \pi^{b,0} - \varkappa^{\mu i} \partial_i A_\mu^a - \varkappa_i^a \sigma_i + \varkappa_{0k} \pi^{a,k} - \right)$$
 (2.22)

$$-\frac{\varkappa_{0k}\varkappa^{0k}}{\varkappa^{00}}\pi^{a,0}-\varkappa^{0k}\nabla_k^{ab}A_0^b\right)\,,$$

$$\dot{\sigma} = \Pi_i + ig\Gamma^a_{ij}A^a_0\varphi_j. \tag{2.23}$$

The canonical variables obey the usual equal-time commutation relations

$$\left[A^{a}_{\mu}, \pi^{b,\nu}\right] = i\delta^{ab}\delta^{\mu}_{\nu}, \ \left[\sigma_{i}, \Pi_{j}\right] = i\delta_{ij}, \ \left[\psi_{i}, \overline{\Pi}_{j}\right] = i\delta_{ij}. \tag{2.24}$$

To find the canonical momenta conjugate to the fictitious fields, as well as the corresponding anticommutation relations, one can use Schwinger's action principle [6]; see Appendix A, where it has also been demonstrated that these expressions are, in fact, the only possible ones. As a result, we find

$$\Pi_C^a = -\varkappa^{0\mu} \partial_\mu C^{+a} \,, \ \Pi_{C^+}^a = \varkappa^{0\mu} \nabla_\mu^{ab} C^b \,, \tag{2.25}$$

$$\dot{C}^{+a} = -\frac{1}{\varkappa^{00}} \left(\Pi_C^a + \varkappa^{0k} \partial_k C^{+a} \right) ,$$

$$\dot{C}^a = \frac{1}{\varkappa^{00}} \left(\Pi_{C^+}^a - g \varkappa^{00} f^{abd} A_0^b C^d - \varkappa^{0k} \nabla_k^{ab} C^b \right) , \qquad (2.26)$$

$$\{C^a, \Pi_C^b\} = \{C^{+a}, \Pi_{C^+}^b\} = i\delta^{ab}.$$
 (2.27)

The other anticommutators are equal to zero. The corresponding Hamiltonian reads

$$H = \pi^{a,\mu} \dot{A}^{a}_{\mu} + \Pi_{i} \dot{\sigma}_{i} + \overline{\Pi}_{i} \dot{\psi}_{i} + \Pi_{C}^{a} \dot{C}^{a} + \Pi_{C}^{a} \dot{C}^{+a} - L =$$

$$= \frac{1}{2 \left(\varkappa^{00}\right)^{2}} \pi^{a,0} \left(\alpha^{ab} - \delta_{ab} \varkappa_{0k} \varkappa^{0k}\right) \pi^{b,0} - \frac{1}{2} \pi^{a,k} \pi_{a,k} + \pi^{a,k} \nabla_{k}^{ab} A_{0}^{b} +$$

$$+ \frac{1}{\varkappa^{00}} \pi^{a,0} \left(\varkappa^{0k} \pi_{k}^{a} - \varkappa^{i\nu} \partial_{i} A_{\nu}^{a} - \varkappa^{0k} \nabla_{k}^{ab} A_{0}^{b} - \varkappa_{i}^{a} \sigma_{i}\right) + \frac{1}{2} \Pi_{i}^{2} +$$

$$+ ig \Pi_{i} \Gamma_{ik}^{a} A_{0}^{a} \varphi_{k} + \frac{1}{4} G_{ik}^{a} G^{a,ik} - \frac{1}{2} \left(\partial_{k} - ig \Gamma^{a} A_{k}^{a}\right) \varphi \cdot \left(\partial^{k} - ig \Gamma^{b} A^{b,k}\right) \varphi -$$

$$- \overline{\psi} \gamma^{k} \left(i \partial_{k} + g \tau^{a} A_{k}^{a}\right) \psi - g \overline{\psi} \gamma^{0} \tau^{a} A_{0}^{a} \psi + \frac{1}{\varkappa^{00}} \Pi_{C}^{a} \Pi_{C}^{a} -$$

$$- \frac{1}{\varkappa^{00}} \Pi_{C}^{a} \varkappa^{0k} \nabla_{k}^{ab} C^{b} - \frac{1}{\varkappa^{00}} \Pi_{C}^{a} \varkappa^{0k} \partial_{k} C^{+a} - g \Pi_{C}^{a} f^{abd} A_{0}^{b} C^{d} +$$

$$+ \partial_{i} C^{+a} \left(\varkappa^{ij} - \frac{\varkappa^{0i} \varkappa^{0j}}{\varkappa^{00}}\right) \nabla_{j}^{ab} C^{b} - ig C^{+a} \varkappa_{i}^{a} \Gamma_{ij}^{b} \varphi_{j} C^{b} . \tag{2.28}$$

One can easily see that the canonical equations that follow from Hamiltonian (2.28) are identical with the Lagrangian equations. For the fictitious particles, this has been verified in Appendix A.

We need an expression for $\dot{\pi}^{a,0}$ in terms of the canonical coordinates and momenta. It can be found with the help of the following relation, that holds true for any operator Q:

$$\dot{Q} = i[H, Q] . \tag{2.29}$$

We have

$$\dot{\pi}^{a,0} = -\frac{1}{\varkappa^{00}} \varkappa^{0i} \partial_i \pi^{a,0} - \frac{1}{\varkappa^{00}} \varkappa^{0i} \nabla_i^{ab} \pi^{b,0} + \nabla_k^{ab} \pi^{b,k} - ig\Pi_i \Gamma_{ij}^a \varphi_j + g\overline{\psi} \gamma^0 \tau^0 \psi + g\Pi_C^b f^{bad} C^d.$$
(2.30)

From (2.16), which holds identically, it follows that on the equations of motion relation (2.16) is valid also for the quantity $L - L_0$, that is,

$$\left(\nabla_{\mu}^{ab}\varkappa^{\mu\nu}\partial_{\nu} + ig\varkappa_{i}^{b}\Gamma_{ij}^{a}\varphi_{j}\right)\alpha_{bc}t^{c} \equiv t^{b}\alpha_{bc}T^{ca} =
= -g\varkappa^{\mu\nu}\partial_{\mu}\left(\partial_{\nu}C^{+b}f^{bad}C^{d}\right) - g^{2}\hat{A}_{\mu}^{ab}\varkappa^{\mu\nu}\partial_{\nu}C^{+d}f^{dbf}C^{f} +
+ g^{2}C^{+b}\varkappa_{i}^{b}\Gamma_{ij}^{d}\Gamma_{jk}^{a}\varphi_{k}C^{d}, \,\,\hat{A}_{\mu}^{ab} = f^{acb}A_{\mu}^{c},$$
(2.31)

where T^{ab} stands for the operator

$$T^{ab} = \partial_{\mu} \varkappa^{\mu\nu} \nabla^{ab}_{\nu} + ig \varkappa^{a}_{i} \Gamma^{b}_{ij} \varphi_{j}. \tag{2.32}$$

The equations of motion for the fictitious fields read as follows:

$$T^{ab}C^b = C^{+b}T^{ba}. (2.33)$$

3 Ward identities

In this section, we deduce the Ward identities by using only the equations of motion and canonical commutation relations.

Consider the vacuum mean value

$$Z_{C^{d}C^{+a}} \equiv \left\langle 0 \left| TC^{d}(y) C^{+a}(x) \exp(iQ) \right| 0 \right\rangle, \tag{3.1}$$

where

$$Q = \int du \, Q(u) = \int du \, \left(J^{b,\mu} A^b_{\mu} + J_i \varphi_i + \overline{\eta} \psi + \overline{\psi} \eta \right) \,. \tag{3.2}$$

An arbitrary operator P(z) obeys the relation

$$\partial_{0}^{(z)} \langle P(z) \rangle = \left\langle \dot{P}(z) \right\rangle + i \int du \, \delta(z_{0} - u_{0}) \, \langle [P(z), Q(u)] \rangle +$$

$$+ \left\langle 0 \left| T \left\{ \exp(iQ) \, \delta(z_{0} - y_{0}) \left[P(z), C^{d}(y) \right] C^{+a}(x) \right. +$$

$$+ \left. C^{d}(y) \, \delta(z_{0} - x_{0}) \left[P(z), C^{+a}(x) \right] \right\} \left| 0 \right\rangle,$$

$$(3.3)$$

where the following notation has been used:

$$\langle P(z) \rangle \equiv \langle 0|TP(z)C^{d}(y)C^{+a}(x)\exp(iQ)|0\rangle. \tag{3.4}$$

Because of the fact that all the fields (anti)commute at equal times, we have

$$t^{f}\left(z\right)Z_{CC^{+}} = \left\langle t^{f}\left(z\right)\right\rangle. \tag{3.5}$$

Expressions of the form $P(A, \varphi, \psi, \overline{\psi}) Z_{CC^+}$ imply that the function P is subject to the replacement

$$A \to \frac{1}{i} \frac{\delta}{\delta J}$$
, etc. (3.6)

In particular,

$$t^{f}\left(z\right)Z_{CC^{+}} = \varkappa^{\mu\nu}\partial_{\mu}^{(z)}\left\langle A^{f}\left(z\right)\right\rangle + \varkappa_{i}^{f}\left\langle \varphi_{i}\left(z\right)\right\rangle. \tag{3.7}$$

Further, relations (2.17)-(2.27), (2.30), (3.3) lead to

$$\dot{t}^{f}(z) Z_{CC^{+}} = \langle \dot{t}^{f}(z) \rangle + \frac{1}{\varkappa^{00}} \alpha^{fb} J^{b,0}(z) Z_{CC^{+}}, \qquad (3.8)$$

$$\ddot{t}^{f}(z) Z_{CC^{+}} = \langle \ddot{t}^{f}(z) \rangle + \frac{1}{\varkappa^{00}} \alpha^{fb} \dot{J}^{b,0}(z) Z_{CC^{+}} - \frac{1}{\varkappa^{00}} \alpha^{fb} \langle Q_{R}^{b}(z) \rangle - \frac{1}{\varkappa^{00}} \alpha^{fb} \langle \left[\frac{1}{\varkappa^{00}} \varkappa^{0i} \partial_{i} Z^{b,0}(z) + \frac{1}{\varkappa^{00}} \varkappa^{0i} \nabla_{i}^{bc} J^{c,0}(z) + \nabla_{0}^{bc} J^{c,0}(z) \right] \rangle + \delta(z - y) \frac{1}{\varkappa^{00}} \alpha^{fb} f^{bdd'} Z_{C^{d'}C^{+a}}. \qquad (3.9)$$

In (3.9), the expression Q_R denotes an infinitesimal variation of the term with the sources,

$$Q_{R}^{b}(z) = -\nabla_{\mu}^{bc} J^{c,\mu}(z) + igJ_{i}(z) \Gamma_{ij}^{b} \varphi_{j}(z) + ig\overline{\eta}(z) \tau^{b} \psi(z) - ig\overline{\psi}(z) \tau^{b} \eta(z).$$

$$(3.10)$$

Relations (3.8) and (3.9) allow one to obtain the following:

$$t^{f}(z) \alpha_{fc} T^{cb} Z_{CC^{+}} = \left\langle t^{f}(z) \alpha_{fc} T^{cb} \right\rangle - \left\langle Q_{R}^{b}(z) \right\rangle + i\delta(z - y) f^{bdd'} Z_{C^{d'}C^{+d}}.$$

$$(3.11)$$

We now use relation (2.31) in the first summand of the r.h.s. of (3.11), which is a valid operation, since all the derivatives are already under the symbol of T-product; we assume b = d, y = z and take an integral over y, as well as a sum over d, namely,

$$\int dy \, t^{f}(y) \, \alpha_{fc} T^{cd} Z_{CC^{+}} = -\int dy \, \left\langle Q_{R}^{d}(y) \right\rangle +
+ \int dy \, \left\langle \left[-g \varkappa^{\mu\nu} \partial_{\mu} \left(\partial_{\nu} C^{+b}(y) \, f^{bdf} C^{f}(y) \right) - g^{2} \hat{A}_{\mu}^{db}(y) \, \varkappa^{\mu\nu} \partial_{\nu} C^{+f}(y) \, f^{fbn} C^{n}(y) +
+ g^{2} C^{+b}(y) \, \varkappa_{i}^{b} \Gamma_{ij}^{f} \Gamma_{jk}^{d} \varphi_{k}(y) \, C^{f}(y) \right] \right\rangle.$$
(3.12)

Using the equations of motion for C and C^+ (2.33), as well as the anticommutativity of the operators C, one can prove (see Appendix B) that the second summand in the r.h.s. of (3.12) is equal to zero. Then (3.12) takes the form

$$\int dy \, t^b(y) \, \alpha_{bc} T^{cd} Z_{CC^+} = -\left\langle \int dy Q_R^d(y) \right\rangle. \tag{3.13}$$

In view of expression (2.32), one can easily see that the derivatives in all the terms T^{cd} commute with the symbol of T-product, with the exception of the term $\partial_0 \varkappa^{0\nu} \nabla_{\nu}$. By virtue of (2.33), (2.25), (2.27), we find

$$T^{cd}Z_{CC^{+}} = i\delta^{cd}\delta(y-z)Z, \qquad (3.14)$$

where Z stands for the vacuum mean value

$$Z \equiv \langle 0 | T \exp(iQ) | 0 \rangle . \tag{3.15}$$

As a result, the Ward identity for the function Z takes the form

$$\alpha_{ab}t^{b}\left(x\right) = i\left\langle 0\left|T\int dyQ_{R}^{b}\left(y\right)C^{b}\left(y\right)C^{+a}\left(x\right)\right|0\right\rangle. \tag{3.16}$$

In order to present (3.16) in the usual form, we utilize a relation that follows from (3.14),

$$P(A, \varphi, \psi) Z_{CC^{+}} = i D^{da}(y, x) P(A, \varphi, \psi) Z, T^{ab} D^{bc} = \delta^{ac},$$
 (3.17)

for an arbitrary function P. This relation allows one to present (3.16) in the usual form which can be found in the literature:

$$\left[\alpha_{ab}t^{b}\left(x\right) + \int dy \, Q_{R}^{b}\left(y\right) D^{ba}\left(y,x\right)\right] Z = 0. \tag{3.18}$$

For the first time, an identity of the form (3.18) has been obtained by Fradkin [7] for an Abelian theory, as well as by Slavnov [8] and Taylor [9] for a non-Abelian gauge theory.

Identity (3.16) has a simple meaning. The substitution of a new field t(x) into this Green function is equivalent to the sum (over the number of fields in the initial Green function) of Green's functions that do not contain the field t(x) and are deduced from the initial Green function with the help of an infinitesimal gauge transformation (2.13)–(2.15) of one of the fields with the gauge function $\Lambda \sim CC^+$. In case one of the fields in the Green function is a gauge-invariant operator, an insertion of the field t(x) into such a function yields zero.

4 Gauge-dependence of Green's functions

In this section, we find a relation between Green's functions in the gauges α_{ab} , $\varkappa^{\mu\nu}$, \varkappa^a_i and

$$\overline{\alpha}_{ab} = \alpha_{ab} + \delta \alpha_{ab} \,, \, \overline{\varkappa}^{\mu\nu} = \varkappa^{\mu\nu} + \delta \varkappa^{\mu\nu} \,, \, \overline{\varkappa}^a_i = \varkappa^a_i + \delta \varkappa^a_i \,.$$

As a preliminary step, we make the following remark. In the framework of canonical formalism, the generating functionals (3.1), (3.15), or, equivalently, the Green function, are computed by making use of the vertices determined by H_{int} and also with the help of non-covariant propagators, being the actual vacuum mean values of the T-products of free fields.

In case the Hamiltonian is quadratic in its momenta, one can pass (due to Wick) to an effective diagrammatic technique,² in which the vertices are determined by $-L_{\rm int}$; the field propagators are determined by Λ^{-1} (Λ being a differential operator that enters the free Lagrangian $L \sim \frac{1}{2}\varphi\Lambda\varphi$), and, besides, there may also appear some additional vertices.

The additional vertices are determined by the matrix present in those summands of the Hamiltonian that are quadratic in momenta [9]. In the case under consideration, the effective Lagrangian that determines the additional diagrams equals to

$$-\frac{i}{2}\delta\left(0\right)\operatorname{Sp}\,\ln\alpha_{ab}\,,\tag{4.1}$$

i.e., there are no additional diagrams. Thus, the generating functionals (3.1), (3.15) can be presented in the form

$$Z = \exp\left(\frac{1}{2}\delta\left(0\right)\operatorname{Sp}\,\ln\alpha_{ab}\right)Z_{W}\,,\tag{4.2}$$

$$Z_{CC^{+}} = \exp\left(\frac{1}{2}\delta\left(0\right)\operatorname{Sp}\,\ln\alpha_{ab}\right)Z_{CC^{+}W},\,\,(4.3)$$

where Z_W and Z_{CC^+W} are now computed with the help of the usual Feynman diagrams.

It is easy to demonstrate (Appendix C) that the variation of Green's functions with respect to the parameters entering the Lagrangian L is given by an insertion of the "field" $i \int dx \, \delta L$ into Green's functions. Due to the fact that the vacuum mean value ξ depends on the gauge parameters, we have

$$\delta Z = i \int dx \, \overline{\delta} L Z + i \int dx \, \delta \xi_i \left(\frac{\delta L}{\delta \sigma_i} + \frac{\delta Q}{\delta \sigma_i} \right) Z + \frac{1}{2} \delta \left(0 \right) \alpha^{ab} \delta \alpha_{ab} Z \,, \tag{4.4}$$

²One can pass to this diagrammatic technique also in the general case. However, in case the Hamiltonian is more than quadratic in its momenta, one cannot find the additional diagrams in a manifest form.

where $\overline{\delta}L$ denotes a variation of the Lagrangian with respect to those gauge parameters that enter the Lagrangian in a manifest way:

$$\int dx \, \overline{\delta} L = i \int dx \, \left[t^a \left(\frac{1}{2} \alpha^{bc} \delta \alpha_{cd} t^d + \delta t^b \right) \alpha_{ab} + C^{+a} \delta T^{ab} C^b \right]. \tag{4.5}$$

The second term in (4.4) equals to zero, because it represents one of the equations satisfied by Z as a function of the sources. This fact follows from the Euler equations for the fields, as well as from definition (3.15) and from the canonical commutation relations.

Let us now use the Ward identity in the form (3.16):

$$\delta Z = -\int dx \left(\frac{1}{2} \alpha^{ac} \delta_{cb} t^b(x) + \delta t^a(x) \right) \left\langle 0 \left| T \exp(iQ) \int dy \, Q_R^d(y) \, C^d(y) \, C^{+a}(x) \right| 0 \right\rangle +$$

$$+ i \left\langle 0 \left| T \exp(iQ) \int dx \, C^{+a}(x) \, T^{ab} C^b(x) \right| 0 \right\rangle + \frac{1}{2} \delta(0) \, \alpha^{ab} \delta \alpha_{ab} Z \,.$$

$$(4.6)$$

With allowance for definitions (2.5), (2.32) and (3.10), the sum of the second term and the part of the first term which contains δt in the r.h.s. of (4.6) yields

$$-\left\langle 0\left|T\exp\left(iQ\right)\int dxdy\,Q_{R}^{a}\left(x\right)C^{a}\left(x\right)C^{+b}\left(y\right)\delta t^{b}\left(y\right)\right|0\right\rangle .\tag{4.7}$$

Let us recall that we compute Z with the help of Feynman diagrams, so that the time derivatives of t, δt , δT and Q_R should be regarded as commuting with the symbol of T-product. However, relation (3.17) remains valid. Due to this fact, we finally have

$$\delta Z = -i \int dx dy \, Q_R^a(x) \, D^{ab}(x, y) \, \overline{\Lambda}^b(y) \, Z \,, \tag{4.8}$$

$$\Lambda^{a}(x) = \frac{1}{2} \alpha^{ab} \delta \alpha_{bc} t^{c}(x) + \delta t^{c}(x) . \tag{4.9}$$

Relation (4.8) has a simple meaning as well: an infinitesimal change of the gauge parameters in the Green function is equivalent to an infinitesimal gauge transformation of each of the fields with the gauge parameter $\Lambda \sim D\overline{\Lambda}$. In particular, (non-renormalized) Green's functions of gauge-invariant operators are gauge-independent.

Concluding this section, notice that one could remain in the framework of the Dyson T-product and use the variation δZ (Appendix C) in the form

$$\delta Z = -i \left\langle 0 \left| T \int dx \, \delta H \exp(iQ) \right| 0 \right\rangle.$$

This would only lead to some more tedious calculations, that would naturally leave relation (4.8) unaltered.

5 Gauge invariance of partition function

In this section, we examine the problem of defining the partition function in gauge theories. Notice that if one defines the partition function as usual,

$$Z = \operatorname{Sp} e^{-\beta H} \tag{5.1}$$

[where Sp is defined (in a space with indefinite metric) so that a cyclic permutation of operators under the symbol of Sp is admissible], then it is not gauge-invariant, since this case does not comply with the Ward identities (3.16), which provide a basis for the gauge-invariance of physical quantities.

Indeed, in the simplest case of free electromagnetic field, described by the Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}^{2} - \partial_{\mu}C^{+}\partial^{\mu}C + \frac{\alpha}{2}(\partial_{\mu}A^{\mu})^{2}, \qquad (5.2)$$

identity (3.16) for the temperature propagator of the field A_{μ} takes the form

$$\alpha \partial^{\mu} \langle A_{\mu} A_{\nu} \rangle = -\partial_{\nu} \langle CC^{+} \rangle , \qquad (5.3)$$

where we have introduced the notation

$$\langle Q \rangle = \operatorname{Sp} e^{-\beta H} T_{\tau} Q \tag{5.4}$$

for any operator depending on the "temperature time" τ [10]. The right- and left-hand sides of relation (5.3) can be computed directly, and, in both cases, they are given by the kernel $\partial_{\nu} \frac{1}{\Box}$. Nevertheless, the Fourier-image of the l.h.s is known to contain only even-valued frequencies, whereas the Fourier-image of the r.h.s., just as Green's functions of any Fermionic operators, contains only odd-valued frequencies (see also Appendix D). Therefore, relation (5.3) does not take place.

Since the operator structure and diagrammatic technique for statistical temperature Green's functions are known [10] to be completely analogous to the corresponding expressions of field theory, the derivation of the Ward identities and gauge properties for temperature functions should be carried out in complete analogy with the corresponding calculations of Section 4.

There is, however, an operation that should be examined in more detail. This is integration by parts, which has been used several times in Section 4 (see below). Integration by parts is valid in case the corresponding vertex conserves momentum (i.e., the sum of the momenta of all the fields in a given vertex equals to zero). Within the temperature techniques, this condition holds true only in case the sum of frequencies of all the fields in a vertex is even-valued [10]. As regards the problem we discuss here, the "criminal" cases of integration by parts arise as one passes from (3.12) to (3.13), that is, as one shifts the action of a derivative from the field t to the field t in the l.h.s. of (3.12), and also as one proves the fact that the second term in the r.h.s. of (3.12) is equal to zero. Given this, one encounters vertices of the kind

$$\int dy \, \partial t \, C \,, \quad \int dy \, C^+ CC \,, \quad \int dy \, C^+ CA \,, \text{ etc} \,. \tag{5.5}$$

One can observe that integration by parts is valid in case the fictitious Fermionic field C (as well as the field A) contains only even-valued frequencies.

Thus, the partition function defined by formula (5.4) is not gauge-invariant. In order to provide gauge-invariance, it is also necessary that the Green function of a fictitious Fermionic field should contain, nevertheless, only even-valued frequencies. Besides, the Wick theorem must also be valid, thus making it possible to deduce relation (4.4); see also Appendix C. Each of these conditions can be fulfilled.

Let us recall that the operator

$$S = e^{-\beta H + \beta \mu_i N_i} T_\tau \exp\left(Q\right) \,, \tag{5.6}$$

$$Q = \int_0^\beta d\tau \int d^3x \left(J^{a,\mu} A^a_\mu + J_i \varphi_i + \overline{\eta} \psi + \overline{\psi} \eta \right)$$
 (5.7)

can be presented in the form

$$S = e^{-\beta H + \beta \mu_i N_i} T_{\tau} \exp\left(Q^{(0)} - \int_0^{\beta} d\tau \int d^3 x \left(H_{\text{int}}^{(0)}\right)\right).$$
 (5.8)

The dependence of the Heisenberg operators in (5.6) on the temperature parameter τ is given by the equation

$$\frac{\partial}{\partial \tau} A^a_{\mu} = \left[\left(H - \mu_i N_i \right), A^a_{\mu} \right]. \tag{5.9}$$

For any other operators, it is determined in a similar way and is given by a formal replacement $it \to \tau$. In (5.8), each of the operators is free, and its dependence on τ is determined by the equation

$$\frac{\partial}{\partial \tau} A^a_{\mu} = \left[\left(H_0 - \mu_i N_i \right), A^a_{\mu} \right]. \tag{5.10}$$

We have also introduced some terms with chemical potentials (in case they are necessary), assuming that N_i are gauge-invariant.

Consider some matrix element S (which will be denoted by $\langle \rangle$). It can be presented in the form

$$\langle S \rangle = \exp\left(-H_{\text{int}}\right) \langle S \rangle_0 |_{\bar{J}_{\mu}^a = \bar{J}_i = \theta^a = \theta^{+a} = \bar{\theta}^a = \bar{\theta}^{+a} = 0} . \tag{5.11}$$

In (5.11), it is implied that the operator acting on $\langle S \rangle_0$ is subject to the replacement

$$A^a_\mu \to \frac{\delta}{\delta J^a_\mu}$$
, etc, (5.12)

and the function $\langle S \rangle_0$ is defined as follows:

$$\langle S \rangle_0 = \left\langle e^{-\beta H_0 + \beta \mu_i N_i} T_\tau \exp\left(Q^{(0)} + \overline{Q}^{(0)}\right) \right\rangle, \tag{5.13}$$

$$\overline{Q} = \int_0^\beta d\tau \int d^3x \left(\overline{J}_\mu^a \pi^{a,\mu} + \overline{J}_i \Pi_i + \theta^{+a} C^a + C^{+a} \theta^a + \overline{\theta}^{+a} \Pi_{C^+}^a + \Pi_C^a \overline{\theta}^a \right). \tag{5.14}$$

All the operators in (5.13) are free. The matrix elements (5.11), (5.13) obey the Wick theorem provided that the free Green functions determined by (5.13) [i.e., the coefficients of the series expansion of (5.13) in the powers of sources] should decompose into a product of two-point Green's functions. In other words, expression (5.13) as a function of sources should be given by an exponential of a quadratic form with respect to the sources. The following matrix element possesses the required property:

$$\langle S \rangle_0 = \sum_n \prod_i \lambda_i^{n_i} \left\langle n \left| e^{-\beta H_0 + \beta \mu_i N_i} T_\tau \exp\left(Q^{(0)} + \overline{Q}^{(0)}\right) \right| n \right\rangle, \tag{5.15}$$

where

$$|n\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \dots \tag{5.16}$$

 $|n_i\rangle$ are n-particle states, normalized by ± 1 , of the particles of i-th type, being eigenstates of the free Hamiltonian H_0 and of the Hamiltonian $H_0 - \mu_i N_i$; the numbers λ_i depend on the type of a particle (in principle, they may also depend on the momentum of a particle). In Appendix D, it is shown that (5.15) equals to

$$\langle S \rangle_0 = Z_0^{(\lambda_i)} \exp\left(\frac{1}{2}J^\alpha D_{\alpha\beta}^H J^\beta\right) \exp\left(\xi_i J_i\right) ,$$
 (5.17)

where

$$Z_0^{(\lambda_i)} = \langle S \rangle_0 |_{J^{\alpha} = 0} , \qquad (5.18)$$

while J^{α} denotes the set of all the sources:

$$J^{\alpha} = \left\{ \overline{J}_{\mu}^{a}, J^{a,\mu}, \overline{J}_{i}, J_{i}, \eta, \overline{\eta}, \overline{\theta}^{a}, \theta^{a}, \overline{\theta}^{+a}, \theta^{+a} \right\},$$

$$(5.19)$$

and $D^H_{\alpha\beta}$ is identical with the two-point Green function of free fields:

$$D_{\alpha\beta}^{H}(x_{1}, x_{2}) = \frac{1}{Z_{0}^{(\lambda_{i})}} \sum_{n} \prod_{i} \lambda_{i}^{n_{i}} \left\langle n \left| e^{-\beta H_{0} + \beta \mu_{i} N_{i}} T_{\tau} \omega_{\alpha}^{(0)}(x_{1}) \omega_{\beta}^{(0)}(x_{2}) \right| n \right\rangle.$$
 (5.20)

 ω_{α} is the set of all the field operators:

$$\omega_{\alpha} = \left\{ A_{\mu}^{a}, \, \pi^{a,\mu}, \, \sigma_{i}, \, \Pi_{i}, \, \psi, \, \overline{\psi}, \, C^{a}, \, \Pi_{C}^{a}, \, C^{+a}, \, \Pi_{C+}^{a} \right\}.$$
 (5.21)

x is the set of the coordinates: $x = (\tau, \vec{x})$.

Using the equations of motion and equal-time commutation relations for $\omega_{\alpha}^{(0)}$, one can easily verify that $D_{\alpha\beta}^H$ has the form

$$\Lambda^{H}_{\alpha\beta}D^{H}_{\beta\gamma} = -\delta_{\alpha\gamma} \,. \tag{5.22}$$

See the definition of $\Lambda^H_{\alpha\beta}$ in Appendix C. Thus, the quantity

$$M^{(\lambda_i)} = \frac{1}{Z_0^{(\lambda_i)}} \langle S \rangle \tag{5.23}$$

has a typical structure of a generating functional in quantum field theory; namely, its calculation can be carried out by the same diagrammatic technique, and it obeys the same functional equations satisfied by a generating functional of quantum field theory. In particular, a literal repetition of the reasonings that are presented in quantum field theory shows [9] that $M^{(\lambda_i)}$ obeys the relation

$$M^{(\lambda_i)} = e^{\frac{1}{2}\delta(0)\operatorname{Sp}\ln\alpha_{ab}}M_W^{(\lambda_i)}, \qquad (5.24)$$

where $M_W^{(\lambda_i)}$ is computed by the "Wick" rules

$$M_W^{(\lambda_i)} = \exp\left(L_{\text{int}}\right) \exp\left(\frac{1}{2} J^{\alpha} D_{\alpha\beta}^L J^{\beta}\right) \exp\left(\xi_i J_i\right) . \tag{5.25}$$

In (5.25), the source J^{α} is introduced only for the fields (that is, $\overline{J}_{\mu}^{a} = \overline{J}_{i} = \overline{\theta}^{a} = \overline{\theta}^{+a} = 0$), while the Green function of the fields satisfies the relation

$$\Lambda_{\alpha\beta}^{L}D_{\beta\gamma}^{L} = -\delta_{\alpha\gamma} \tag{5.26}$$

(the definition of $\Lambda_{\alpha\beta}^L$ is given in Appendix C). Besides, in the course of taking a variation of gauge parameters, $M^{(\lambda_i)}$ changes as follows:

$$\delta M^{(\lambda_i)} = -e^{-H_{\rm int}} \int_0^\beta d\tau \int d^3x \delta H \exp\left(\frac{1}{2}J^\alpha D^H_{\alpha\beta}J^\beta\right) e^{\xi_i J i} =$$
 (5.27)

$$= -\frac{1}{Z_0^{(\lambda_i)}} \left\langle e^{-\beta H_0 + \beta \mu_i N_i} T_\tau \int_0^\beta d\tau \int d^3x \delta H \exp(Q) \right\rangle, \tag{5.28}$$

$$\delta M_W^{(\lambda_i)} = e^{L_{\text{int}}} \int_0^\beta d\tau \int d^3x \, \delta L \exp\left(\frac{1}{2} J^\alpha D_{\alpha\beta}^L J^\beta\right) e^{\xi_i J i} \tag{5.29}$$

(the sources in (5.29) are introduced only for the fields). Relations (5.28) and (5.29) are exact analogues of the corresponding relations (C.9) and (C.5) of quantum field theory. The only difference between $M^{(\lambda_i)}$ and a generating functional of quantum field theory arises if one defines the propagators $D^{(H,L)}$ as the integral operators $\Lambda^{(H,L)-1}$, whose unique definition requires that one should impose certain boundary conditions. The propagators for arbitrary λ_i are computed in Appendix D.

As has been mentioned in the beginning of this section, it is necessary (in order to ensure the possibility of deducing the Ward identities) that the fictitious fields should have only even-valued frequencies. For the remaining fields, we assume the usual relation between statistics and the parity of frequencies. As shown in Appendix D, for all the particles, except the fictitious ones, λ_i must be chosen as follows:

$$\lambda_i = 1$$
 for particles with positive metric,
$$\lambda_i = -1 \text{ for particles with negative metric.}$$
 (5.30)

Notice that such a choice of λ_i ensures the coincidence of a matrix element $\langle \dots \rangle$ with the definition of the trace of an operator (in the subspace of the mentioned particles), so that the relation between the parity of frequencies and the statistics of a field can be found without a manifest calculation of the propagator. For the fictitious particles, the parameters λ_i must be chosen as follows:

$$\lambda_i = -1$$
 for particles with positive metric,
$$\lambda_i = 1 \text{ for particles with negative metric.}$$
 (5.31)

The series expansion of a fictitious field in the powers of creation and annihilation operators is made in Appendix A, where it has been shown that the field C^a contains two particles. Correspondingly, λ_i are equal to

$$\lambda_d = \frac{\varkappa^{00}}{|\varkappa^{00}|}, \ \lambda_b = -\frac{\varkappa^{00}}{|\varkappa^{00}|}.$$
 (5.32)

Notice, once again, that the choice of λ_i for the fictitious particles in accordance with (5.30) implies that their Green function contains only odd-valued frequencies. This is a consequence of the fact that the choice of λ_i according to (5.30) implies that a matrix element $\langle \ldots \rangle$ coincides with the trace of an operator, whereas in this case the propagator of any Fermionic field contains only odd-valued frequencies.

Consequently, given the choice of λ_i in accordance with (5.30), (5.32), a literal repetition³ of the reasonings of Section 3 makes it possible to conclude that $M^{(\lambda_i)}$ obeys the following Ward identity:

$$\alpha_{ab}t^{b}\left(x\right)M^{(\lambda_{i})} = \frac{1}{Z_{0}^{(\lambda_{i})}}\left\langle e^{-\beta H_{0} + \beta \mu_{i}N_{i}}T_{\tau} \int dy Q_{R}^{b}\left(y\right)C^{b}\left(y\right)C^{+a}\left(x\right)\exp\left(Q\right)\right\rangle =$$

$$= i \int dy Q_{R}^{b}\left(y\right)D^{ba}\left(y,x\right)M^{(\lambda_{i})},$$

$$(5.33)$$

³Within the temperature techniques, there remain valid the commutation relations and definitions of canonical momenta (2.17)–(2.27) under the replacement $\partial_t \to i\partial_\tau$. This replacement also implies the coincidence of the equations of motion (2.29) and (5.9), with obvious modifications due to the presence of the operators N_i .

where $\int dy$ denotes

$$\int dy \equiv \int_0^\beta d\tau_y \int d^3y \,. \tag{5.34}$$

Besides, one ought to remember that in terms of the variables τ the D-function is defined by the equation

$$T^{ab}(x) D^{bc}(x,y) = i\delta(\tau_x - \tau_y) \delta(\vec{x} - \vec{y})$$

$$(5.35)$$

and there holds a relation of the form (3.17).

A variation of the gauge parameters changes $M^{(\lambda_i)}$ as follows:

$$\delta M^{(\lambda_i)} = i \int dx \, dy \, Q_R^a(x) \, D^{ab}(x, y) \, \overline{\Lambda}^b(y) \, M^{(\lambda_i)}, \qquad (5.36)$$

where $\overline{\Lambda}$ is given by (4.9).

Thus, the statistical average of gauge-invariant operators, and, in particular, the partition function, are gauge-independent on condition that $Z_0^{(\lambda_i)}$ should be gauge-invariant. To prove the gauge invariance of $Z_0^{(\lambda_i)}$, let us apply the formula

$$\delta e^{-\beta \overline{H}} = -e^{-\beta \overline{H}} \int_{0}^{\beta} d\tau \, e^{\tau \overline{H}} \delta \overline{H} e^{-\tau \overline{H}} \equiv e^{-\beta \overline{H}} T_{\tau} \int_{0}^{\beta} d\tau \, \delta H(\tau) , \qquad (5.37)$$

$$\overline{H} = H_0 - \mu_i N_i \,. \tag{5.38}$$

Thus, the variation of $\ln Z_0^{(\lambda_i)}$ equals to the variation of $M^{(\lambda_i)}$ for the free theory (let us denote the latter by $M_0^{(\lambda_i)}$) with the vanishing sources:

$$\delta \ln Z_0^{(\lambda_i)} = \delta M_0^{(\lambda_i)} \Big|_{I^{\alpha} = 0} . \tag{5.39}$$

From (5.36) it follows that $\ln Z_0^{(\lambda_i)}$ is gauge-invariant. The gauge invariance of $\ln Z_0^{(\lambda_i)}$ can also be established by a direct calculation using the expression for H_0 obtained in Appendices A and E.

In physical gauges (the Coulomb gauge $\varkappa^{0\mu} = \varkappa^a_i = 0$, $\varkappa^{ik} = \delta_{ik}$, $\alpha_{ab} \to \infty$; the unitary gauge $\varkappa^a_i \to \infty$, that corresponds to the gauge $B_m = 0$, with $B_m = \xi^m_i \varphi_i$, being Goldstone Bosons; see, e.g., [11]), non-physical particles are absent, so that the proposed definition of the partition function and statistical average, given by

$$Z = \sum_{n} \prod_{i} \lambda_{i}^{n_{i}} \left\langle n \left| e^{-\beta H + \beta \mu_{i} N_{i}} \left(\dots \right) \right| n \right\rangle , \qquad (5.40)$$

where $|n\rangle$ and λ_i are defined in accordance with (5.16), (5.32), while (...) stands for the T_{τ} -product of operators whose statistical average is to be found, coincides with the usual definition in physical gauges. Since the partition function and statistical average of a gauge-invariant operator are gauge-independent, definition (5.40) yields a correct result in any gauge.

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Appendix

We are now going to deduce the expressions for the canonical momenta conjugate to C and C^+ , as well as for their commutation relations with the help of Schwinger's action principle [6]. Let us present the part of the action that corresponds to the fictitious fields:

$$W = \int_{t_1}^{t_2} dx \left(-\partial_{\mu} C^{+a} \varkappa^{\mu\nu} \nabla^{ab}_{\nu} + igC^{+a} \varkappa^{a}_{i} \Gamma^{b}_{ij} \varphi_{j} C^{b} \right). \tag{A.1}$$

We now find a variation δW :

$$W = \int_{t_1}^{t_2} dx \left(\delta C^{+a} \overrightarrow{T}^{ab} C^b + C^{+a} \overleftarrow{T}^{ab} \Gamma^b_{ij} \varphi_j \delta C^b \right) + G \bigg|_{t_a}^{t_2}, \tag{A.2}$$

where

$$G = \int d^3x \left(-\delta C^{+a} \varkappa^{0\nu} \nabla^{ab}_{\nu} C^b - \varkappa^{0\nu} \partial_{\nu} C^{+a} \delta C^a \right). \tag{A.3}$$

In accordance with Schwinger's action principle, G is the generator of variations of the field variables at a fixed moment of time:

$$\delta C^a = i [G, C^a], \ \delta C^{+a} = i [G, C^{+a}].$$
 (A.4)

Then, introducing notation (2.25), we deduce from (A.3) and (A.4) that the anticommutation relations (2.27) hold true, whereas the anticommutators between Π_C and Π_{C^+} , C and C^+ are equal to zero.

Let us now prove the fact that definition (2.25) is, in a certain sense, unique. Namely, we define the canonical momenta as

$$\Pi_C^a = -\alpha \varkappa^{0\mu} \partial_\mu C^{+a} \,, \,\, \Pi_{C^+}^a = \beta \varkappa^{0\mu} \partial_\mu C^b \tag{A.5}$$

and suppose that (2.27) holds true. We then construct the Hamiltonian

$$H = \alpha_1 \Pi_C \dot{C} + \beta_1 \Pi_C \dot{C}^+ - L = \frac{1}{\varkappa^{00}} \left(\frac{\alpha_1}{\beta} + \frac{\beta_1}{\alpha} - \frac{1}{\alpha \beta} \right) \Pi_C^a \Pi_{C^+}^b - \frac{\alpha_1}{\varkappa^{00}} \Pi_C^a \left(g \varkappa^{00} f^{abd} A_0^b + \varkappa^{0k} \nabla_k^{ad} \right) C^d + \partial_k C^{+a} \left(\varkappa^{ik} - \frac{\varkappa^{0i} \varkappa^{0k}}{\varkappa^{00}} \right) \nabla_i^{ab} C^b - \frac{\beta_1}{\varkappa^{00}} \Pi_{C^+}^a \varkappa^{0k} \partial_k C^{+a} - igC^{+a} \varkappa_i^a \Gamma_{ij}^b \varphi_j C^b \,. \tag{A.6}$$

Let us now demand that the Hamiltonian equations of motion in the form (2.29) for the fields and momenta should yield equations (2.33) and definitions (A.5). We have

$$\dot{C}^{a} = \frac{1}{\varkappa^{00}} \left[\left(\frac{\alpha_{1}}{\beta} + \frac{\beta_{1}}{\alpha} - \frac{1}{\alpha \beta} \right) \Pi_{C^{+}}^{a} - \alpha_{1} \left(g \varkappa^{00} \hat{A}_{0}^{ab} + \varkappa^{0k} \nabla_{k}^{ab} \right) C^{b} \right]. \tag{A.7}$$

By comparison with (A.5), we find

$$\alpha_1 = 1, \ \beta_1 = \frac{1}{\beta}. \tag{A.8}$$

Next,

$$\dot{C}^{+a} = -\frac{1}{\beta \varkappa^{00}} \Pi_C^a - \frac{1}{\beta \varkappa^{00}} \varkappa^{0k} \partial_k C^{+a} , \qquad (A.9)$$

whence

$$\beta = 1 = \beta_1, \ \alpha = \beta. \tag{A.10}$$

Therefore, all the parameters are defined uniquely, and the expressions for Π and H coincide with the corresponding expressions of Section 2. Let us now verify that the Lagrangian equations are also fulfilled. Indeed,

$$\dot{\Pi}_C = -\frac{1}{\varkappa^{00}} \left(g \hat{A}^{ab} + \varkappa^{0k} \nabla_k^{ab} \right) \Pi_C^b + \nabla_i^{ab} \left(\varkappa^{ik} - \frac{\varkappa^{0i} \varkappa^{0k}}{\varkappa^{00}} \right) \partial_k C^{+b} + igC^{+b} \varkappa_i^b \Gamma_{ij}^a \varphi_j , \qquad (A.11)$$

$$\dot{\Pi}_C = -\frac{1}{\varkappa^{0a}} \varkappa^{0k} \partial_k \Pi_C^a - \partial_k \left(\varkappa^{ki} - \frac{\varkappa^{0k} \varkappa^{0i}}{\varkappa^{00}} \right) \nabla_i^{ab} C^b - ig \varkappa_i^a \Gamma_{ij}^b \varphi_j C^b \,. \tag{A.12}$$

Substituting expressions (2.25) into (A.11), (A.12), we find the equations of motion (2.33).

Let us now expand the operator C in the powers of the creation and annihilation operators in the free case:

$$L_0 = -\partial_{\mu} C^{+a} \varkappa^{\mu\nu} \partial_{\nu} C^{+a} + C^{+a} m_{ab} C^b , \qquad (A.13)$$

$$m_{ab} = -g\varkappa_i^a \xi_i^b \,. \tag{A.14}$$

For the sake of simplicity, we suppose that there exists a matrix S such that transforms the matrix m to a diagonal form:

$$S_{ab}^{-1} m_{bc} S_{cd} = \mu_a \delta_{ad} \,. \tag{A.15}$$

Introducing the fields

$$V_a^+ = C^{+b} S_{ba} \,, \ V_a = S_{ab}^{-1} C^b \,,$$
 (A.16)

we bring Lagrangian (A.13), as well as the equations of motion and commutation relations, to the form

$$L_0 = -\partial_\mu V_a^+ \varkappa^{\mu\nu} \partial_\nu V_a + \mu_a V_a^+ V_a \,, \tag{A.17}$$

$$\left(\varkappa^{\mu\nu}\partial_{\mu}\partial_{\nu} + \mu_{a}\right)V_{a} = \left(\varkappa^{\mu\nu}\partial_{\mu}\partial_{\nu} + \mu_{a}\right)V_{a}^{+} = 0, \tag{A.18}$$

$$\left\{ \varkappa^{0\nu} \partial_{\nu} V_a , V_b \right\} = -\left\{ \varkappa^{0\nu} \partial_{\nu} V_a^+ , V_b \right\} = i\delta_{ab} , \tag{A.19}$$

while the remaining anticommutators are equal to zero.

From (A.18) it follows that the expansion of the field V_a has the from

$$V_{a}\left(x\right) = \int \frac{dp}{\left(2\pi\right)^{3/2} \sqrt{|\varkappa^{00}|} \sqrt{\frac{2}{\varkappa^{00}} \varkappa^{0\nu} p_{\nu}}} \left[e^{-ipx} d_{a}\left(p\right) \delta\left(p_{0} - \omega_{p}^{a}\right) + b_{a}^{+}\left(p\right) e^{ipx} \delta\left(p_{0} - \Omega_{p}^{a}\right)\right], \quad (A.20)$$

$$\omega_p^a = -\frac{1}{\varkappa^{00}} \varkappa^{0k} p_k + \left[\frac{1}{\varkappa^{00}} \left(\frac{\varkappa^{0i} \varkappa^{0k}}{\varkappa^{00}} - \varkappa^{ik} \right) p_i p_k + \frac{\mu_a}{\varkappa^{00}} \right]^{1/2} ,$$

$$\Omega_p^a = \frac{1}{\varkappa^{00}} \varkappa^{0k} p_k + \left[\frac{1}{\varkappa^{00}} \left(\frac{\varkappa^{0i} \varkappa^{0k}}{\varkappa^{00}} - \varkappa^{ik} \right) p_i p_k + \frac{\mu_a}{\varkappa^{00}} \right]^{1/2} .$$
(A.21)

Of course, $\varkappa^{\mu\nu}$ and μ_a must be subject to such equations that ω_p^a and Ω_p^a should be real-valued.

With respect to V_a^+ , we assume

$$V_a^+ = (V)_a^+ \ . \tag{A.22}$$

In the general case, C^+ is not Hermitian-conjugate to C. Then, the canonical commutation relations lead to the following rules for the operators d and C:

$$-\left\{d_{a}(p), d_{b}^{+}(q)\right\} = \left\{b_{a}(p), d_{b}^{+}(q)\right\} = \frac{\varkappa^{00}}{|\varkappa^{00}|} \delta_{ab} \delta(\vec{p} - \vec{q}). \tag{A.23}$$

Let us verify, for instance, (A.19), namely,

$$\left\{ \varkappa^{0\nu} \partial_{\nu} V_{a}^{+}(x) , V_{b}(y) \right\} \Big|_{x_{0}=y_{0}} = -\frac{i \delta_{ab}}{2 \left| \varkappa^{00} \right|} \int \frac{\varkappa^{0\nu} p_{\nu} \, dk \, dp}{\left(2\pi \right)^{3} \left[\frac{1}{\varkappa^{00}} \varkappa^{0\nu} p_{\nu} \frac{1}{\varkappa^{00}} \varkappa^{0\mu} p_{\mu} \right]^{1/2}} \times \\
\times \left[e^{-ipx + iky} \left(-\frac{\varkappa^{00}}{\left| \varkappa^{00} \right|} \right) \delta \left(p_{0} - \Omega_{p}^{a} \right) \delta \left(k_{0} - \Omega_{l}^{a} \right) - \\
- \frac{\varkappa^{00}}{\left| \varkappa^{00} \right|} e^{ipx - iky} \delta \left(p_{0} - \omega_{p}^{a} \right) \delta \left(k_{0} - \omega_{k}^{a} \right) \right] \delta \left(\vec{p} - \vec{k} \right) = i \delta_{ab} \delta \left(\vec{x} - \vec{y} \right) . \tag{A.24}$$

The expansion of the initial fields has the form

$$C^{a}(x) = \frac{1}{\sqrt{|\varkappa^{00}|}} \int \frac{dp}{\left[(2\pi)^{3} \frac{2}{\varkappa^{00}} \varkappa^{0\nu} p_{\nu} \right]^{1/2}} S_{ac} \left[e^{-ipx} \delta \left(p_{0} - \omega_{p}^{c} \right) d_{c} \left(p \right) + e^{ipx} \delta \left(p_{0} - \Omega_{p}^{c} \right) b_{c}^{+} \left(p \right) \right] ,$$

$$C^{+a}(x) = \frac{1}{\sqrt{|\varkappa^{00}|}} \int \frac{dp}{\left[(2\pi)^{3} \frac{2}{\varkappa^{00}} \varkappa^{0\nu} p_{\nu} \right]^{1/2}} \left[e^{ipx} \delta \left(p_{0} - \omega_{p}^{c} \right) d_{c}^{+} \left(p \right) + e^{-ipx} \delta \left(p_{0} - \Omega_{p}^{c} \right) b_{c} \left(p \right) \right] S_{ca}^{-1} .$$

$$(A.25)$$

Relation (A.23) shows that the field C contains two kinds of Fermions; one of them has a positive norm, while the other one has an indefinite norm. Of course, this fact is in agreement with the theorem on the relation between spin and statistics.

Let us, finally, present an expression for the Hamiltonian in terms of the creation and annihilation operators:

$$H_{0} = \int d^{3}x \left[\frac{1}{\varkappa^{00}} \Pi_{C}^{a} \Pi_{C^{+}}^{a} - \frac{1}{\varkappa^{00}} \Pi_{C}^{a} \varkappa^{0k} \partial_{k} C^{a} - \frac{1}{\varkappa^{00}} \Pi_{C^{+}}^{a} \varkappa^{0k} \partial_{k} C^{+a} + \right.$$

$$+ \left. \partial_{i} C^{+a} \left(\varkappa^{ij} - \frac{\varkappa^{0i} \varkappa^{0j}}{\varkappa^{00}} \right) \partial_{j} C^{a} + C^{+a} m_{ab} C^{b} \right] =$$

$$= \int d^{3}p \left[-\frac{\varkappa^{00}}{|\varkappa^{00}|} \omega_{p}^{a} d_{a}^{+} (p) d_{a} (p) + \frac{\varkappa^{00}}{|\varkappa^{00}|} \Omega_{p}^{a} b_{a}^{+} (p) b_{a} (p) \right]. \tag{A.27}$$

The propagator of the field C is given by

$$\langle 0 | TC^{a}(x) C^{+b}(y) | 0 \rangle = S_{ac} \langle 0 | TV_{c}(x) V_{d}^{+}(y) | 0 \rangle S_{db}^{-1} =$$

$$= \int \frac{dp}{(2\pi)^{4}} e^{-ip(x-y)} G_{C}^{ab}(p) , \qquad (A.28)$$

$$G_C^{ab}(p) = S_{ad} \frac{-i}{\varkappa^{\mu\nu} p_{\mu} p_{\nu} - \mu_d} S_{db}^{-1} = -i \left(\varkappa^{\mu\nu} p_{\mu} p_{\nu} \delta_{ab} - m_{ab} \right)^{-1} . \tag{A.29}$$

B Appendix

Let us now prove that the second term in the r.h.s. of (3.12) equals to zero.

We use the antisymmetry property for a product of the fields C^a at coincident points, as well as the possibility of freely integrating by parts, since the additional terms arising due to the non-commutativity of ∂_0 and T-product are proportional to $f^{bdf}\delta_{bd}$, which is equal to zero. We then have the following equalities (we imply integration over dy yet do not present it explicitly):

$$-g\partial_{\mu}\varkappa^{\mu\nu}\left(\partial_{\nu}C^{+b}f^{bdf}C^{f}\right)C^{d} = g\varkappa^{\mu\nu}\partial_{\mu}C^{+b}f^{bdf}C^{f}\partial_{\nu}C^{d} =$$

$$= \frac{g}{2}\varkappa^{\mu\nu}\partial_{\mu}C^{+b}f^{bdf}\left(C^{f}\partial_{\nu}C^{d} - \partial_{\nu}C^{d}C^{f}\right) = \frac{g}{2}C^{+b}\overleftarrow{\partial}_{\mu}\varkappa^{\mu\nu}\overleftarrow{\partial}_{\nu}f^{bdf}C^{d}C^{f}, \qquad (B.1)$$

$$-g^{2}f^{dcb}f^{fbn}A^{c}_{\mu}\varkappa^{\mu\nu}\partial_{\nu}C^{+f}C^{n}C^{d} =$$

$$= \frac{g^{2}}{2}\left(f^{dcb}f^{bfn} - f^{ncb}f^{bfd}\right)A^{c}_{\mu}\varkappa^{\mu\nu}\partial_{\nu}C^{+f}C^{n}C^{d} =$$

$$= -\frac{g^{2}}{2}C^{+l}\overleftarrow{\partial}_{\mu}\varkappa^{\mu\nu}f^{lcb}A^{c}_{\nu}f^{bdf}C^{d}C^{f}. \qquad (B.2)$$

Here, we have used the Jacobi identity

$$f^{dcb}f^{bfn} + f^{ncb}f^{bdf} = -f^{fcb}f^{bnd}. (B.3)$$

Next,

$$g^{2}C^{+b}\varkappa_{i}^{b}\Gamma_{ij}^{f}\Gamma_{jk}^{d}\varphi_{k}C^{f}C^{d} = \frac{g^{2}}{2}C^{+b}\varkappa^{b}\left(\Gamma^{f}\Gamma^{d} - \Gamma^{d}\Gamma^{f}\right)\varphi C^{f}C^{d} =$$

$$= i\frac{g^{2}}{2}C^{+l}\varkappa_{k}^{l}\Gamma_{kj}^{b}\varphi_{j}f^{bdf}C^{d}C^{f}. \tag{B.4}$$

In (B.4), we have also used (2.4). Summarizing (B.1), (B.2) and (B.4), we find that the second term in the r.h.s. of (3.12) has the form

$$\frac{1}{2}gC^{+l}\overleftarrow{T}^{lb}f^{bdf}C^{d}C^{f}, \qquad (B.5)$$

which equals to zero owing to the equations (2.33) for C^+ .

C Appendix

Let us now deduce a formula for the variation of Z_W corresponding to a variation of the parameters in the Lagrangian. In the interaction representation, Z_W can be written as follows:

$$Z_W = \langle 0 | T_W \exp\left(iQ + iL_{\text{int}}\right) | 0 \rangle . \tag{C.1}$$

When taking a variation of the parameters in the Lagrangian, we need to examine a variation of the vertices and propagators. According to (C.1), the variation of vertices is given by an insertion into the Green function of a "field" $i \int dx \, \delta L_{\rm int}$.

Since the field propagator equals to

$$D_{ij}^{L} = i \left(\Lambda_{ij}^{L} \right)^{-1} , \qquad (C.2)$$

where Λ^L is a differential operator in the free Lagrangian,

$$L_0 = \frac{1}{2} \varphi_i \Lambda_{ij}^L \varphi_j \,, \tag{C.3}$$

its variation reads

$$\delta D_{ij}^L = D_{ik}^L \left(i\delta \Lambda_{kl}^L \right) D_{lj}^L . \tag{C.4}$$

From (C.4) it follows that the variation of propagators is equivalent to an insertion of the "field" $i \int dx \, \delta L_0$. Thus, the total variation of Green's functions with respect to the parameters of the Lagrangian is given by an insertion of the "field" $i \int dx \, \delta L$, namely,

$$\delta Z_W = \left\langle 0 \left| T_W \left(i \int dx \delta L \right) \exp\left(iQ + iL_{\text{int}} \right) \right| 0 \right\rangle. \tag{C.5}$$

If we do not pass to Wick's rules, Z is given by the formula

$$\delta Z = \langle 0 | T_W \exp(iQ - iH_{\text{int}}) | 0 \rangle . \tag{C.6}$$

where T denotes the symbol of the usual T-ordering. Let us introduce a unifying field, $\omega_{\mu} = (\pi_i, \varphi_i)$. Then (C.6) is computed by the usual Feynman rules with vertices defined by $-iH_{\rm int}$ and by the propagator

$$D_{\mu\nu}^{H} = i \left(\Lambda_{\mu\nu}^{H} \right)^{-1} , \qquad (C.7)$$

with Λ^H being a differential operator in the expression

$$\pi_{i}\dot{\varphi}_{i} - H_{0} = \frac{1}{2}\omega_{\mu}\Lambda_{\mu\nu}^{H}\omega_{\nu} ,$$

$$\Lambda_{\mu\nu}^{H} = \Lambda_{\mu\nu}^{(1)} + \Lambda_{\mu\nu}^{(2)} ,$$
(C.8)

where $\Lambda^{(1)}$ is determined by $\pi_i \dot{\varphi}_i$ and is independent of any parameters; $\Lambda^{(2)}$ is determined by the free Hamiltonian. Next, a literal repetition of the reasonings that have lead us to formula (C.5) yields

$$\delta Z = \left\langle 0 \left| T \left(i \int dx \, \delta H \right) \exp \left(iQ - H_{\text{int}} \right) \right| 0 \right\rangle. \tag{C.9}$$

We also note that a variation of the Lagrangian and that of the Hamiltonian with respect to the parameters are related by

$$-\delta H|_{\pi,\omega} = \delta L|_{\dot{\omega},\omega} , \qquad (C.10)$$

where the notation

$$\delta A|_u$$

stands for a variation of A with a fixed u.

Formulas (C.5) and (C.9) have the same appearance. One must, however, bear in mind that the time derivatives in (C.6) commute with the symbol of T_W -product, whereas in (C.9) the time derivative that arises after the substitution $\pi_i \sim \dot{\varphi}_i$ does not commute with the symbol of T-product. Of course, the final results, calculated by formulas (C.5) and (C.9), are identical, with allowance for relation (4.2).

D Appendix

We are now going to compute the partition functions and Green's functions, as well as to prove Wick's theorem for various free fields with a generalized definition of statistical average.

Let us examine, first of all, the case of one degree of freedom.

A) Bose system:

$$\overline{H} = H_0 - \mu N = \alpha \omega a^+ a, \ \left[a^+, a \right] = \alpha, \ \alpha^2 = 1,$$
(D.1)

$$a(\tau) = e^{-\omega \tau} a, \ a^{+}(\tau) = e^{\omega \tau} a^{+},$$
 (D.2)

We now define a generating functional,

$$Z^{(1)}(J) = \sum_{n} \lambda^{n} \left\langle n \left| e^{-\beta \overline{H}} T_{\tau} \exp \left\{ \int_{0}^{\beta} d\tau \left[J(\tau) a^{+}(\tau) + J^{+}(\tau) a(\tau) \right] \right\} \right| n \right\rangle, \tag{D.3}$$

where

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle , \langle n| = \frac{1}{\sqrt{n!}} \langle 0| a^n, \lambda^2 = 1.$$
 (D.4)

Let us transform the T_{τ} -product in (D.3) to the normal product:

$$T_{\tau} \exp \left\{ \int_{0}^{\beta} d\tau \left(J(\tau) a^{+}(\tau) + J^{+}(\tau) a(\tau) \right) \right\} = \exp \left\{ \int_{0}^{\beta} d\tau_{1} d\tau_{2} J^{+}(\tau_{1}) \Delta(\tau_{1}, \tau_{2}) J(\tau_{2}) \right\} \times \\ \times : \exp \left\{ J_{1} a^{+} + J_{1}^{+} a \right\} :, \tag{D.5}$$

$$\Delta (\tau_1, \tau_2) = \alpha \theta (\tau_1 - \tau_2) e^{-\omega(\tau_1 - \tau_2)}, \qquad (D.6)$$

$$J_{1} = \int_{0}^{\beta} d\tau J(\tau) e^{\omega \tau}, \ J_{1}^{+} = \int_{0}^{\beta} d\tau J^{+}(\tau) e^{-\omega \tau}.$$
 (D.7)

Substituting (D.5) into (D.3), we obtain⁴

$$Z_{1}^{(1)} = e^{J^{+}\Delta J} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \lambda^{n} e^{-\beta \omega n} \frac{J_{1}^{k} J_{1}^{+k}}{n! k! k!} \left\langle n \left| a^{+k} a^{k} \right| n \right\rangle =$$

$$= e^{J^{+}\Delta J} \sum_{k=0}^{\infty} \frac{\left(\alpha J_{1} J_{1}^{+}\right)^{k}}{k! k!} \sum_{n=k}^{\infty} A_{n}^{k} \left(\alpha \lambda e^{-\omega \beta}\right)^{n} =$$

$$= e^{J^{+}\Delta J} \sum_{k=0}^{\infty} \frac{\left(\alpha J_{1} J_{1}^{+}\right)^{k}}{k! k!} x^{k} \frac{d^{k}}{dx^{k}} \sum_{n=0}^{\infty} x^{n} \Big|_{x=\alpha \lambda e^{-\omega \beta}} =$$

$$= \frac{1}{1-x} e^{J^{+}\Delta J} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\left(\alpha x J_{1} J_{1}^{+}\right)^{k}}{k! k!} =$$

$$= \frac{1}{1-x} \exp\left\{ \int_{0}^{\beta} d\tau_{1} d\tau_{2} J^{+} (\tau_{1}) D(\tau_{1}, \tau_{2}) J^{+} (\tau_{2}) \right\}, \qquad (D.8)$$

$$D(\tau_1, \tau_2) = \alpha \theta(\tau_1 - \tau_2) e^{-\omega(\tau_1 - \tau_2)} + \frac{\lambda e^{-\omega \beta - \omega(\tau_1 - \tau_2)}}{1 - \alpha \lambda e^{-\omega \beta}} \equiv \frac{1}{\beta} \sum_n e^{-i\varepsilon_n(\tau_1 - \tau_2)} D_1(\varepsilon_n) , \quad (D.9)$$

$$D_1(\varepsilon_n) = \frac{1}{2} \int_{-\beta}^{\beta} d\tau \, e^{i\varepsilon_n \tau} D_1(\tau) = -\frac{\alpha}{2} \frac{1 + \alpha \lambda (-)^n}{i\varepsilon_n - \omega} \,. \tag{D.10}$$

Next, the partition function $Z_0^{(1)}$ reads

$$Z_0^{(1)} \equiv \sum_n \lambda^n \left\langle n \left| e^{-\beta \overline{H}} \right| n \right\rangle = \frac{1}{1 - \alpha \lambda e^{-\omega \beta}}.$$
 (D.11)

As a result, the generating functional takes the form

$$Z^{(1)}(J) = Z_0^{(1)} \exp\left(\frac{1}{2}J^+D_1J\right),$$
 (D.12)

where D_1 is given by (D.9), (D.10) and equals to

$$D\left(\tau_{1}, \tau_{2}\right) = \frac{1}{Z_{0}^{(1)}} \sum_{n} \lambda^{n} \left\langle n \left| e^{-\beta \overline{H}} T_{\tau} a\left(\tau_{1}\right) a^{+}\left(\tau_{2}\right) \right| n \right\rangle. \tag{D.13}$$

Consequently, we can see that the proposed extension of statistical average obeys Wick's theorem (which is also true for arbitrary λ). Besides, the choice

$$\lambda = \alpha \tag{D.14}$$

implies that the D_1 -function contains only even frequencies. As has been observed in Section 5, in this case there holds the relation

$$\sum_{n} \alpha^{n} \langle n | Q_{1} Q_{2} | n \rangle = \sum_{n} \alpha^{n} \langle n | Q_{2} Q_{1} | n \rangle , \qquad (D.15)$$

being a consequence of the identity

$$1 = \sum_{n} |n\rangle \, \alpha^n \, \langle n| \,\,, \tag{D.16}$$

⁴We remind that $\mathcal{E}_n = \pi n/\beta$.

which makes it possible to find the spectrum of frequencies of the function D_1 by using the Bose-properties of the operators. Nevertheless, the choice $\lambda = -\alpha$ implies that the function D_1 (the Green function of Bose operators) contains only *odd* frequencies. It is interesting that the distribution function in this case is also of the Fermi character:

$$N = \frac{1}{Z_0^{(1)}} \sum_n (-\alpha)^n \langle n | e^{-\beta H} \alpha a^+ a | n \rangle = (e^{\omega \beta} + 1)^{-1}.$$
 (D.17)

In a similar way, we deduce the expression for the generating functional of Green's functions for the field $\varphi(\tau) = \frac{1}{\sqrt{2\omega}} \left(e^{-\omega\tau}a + e^{\omega\tau}a^+\right)$, namely,

$$Z^{(2)} \equiv \sum_{n} \lambda^{n} \left\langle n \left| e^{-\beta \overline{H}} T_{\tau} \exp\left(\int_{0}^{\beta} d\tau J \varphi \right) \right| n \right\rangle = Z_{0}^{(1)} \exp\left(\frac{1}{2} J D_{2} J \right) , \tag{D.18}$$

$$D_{2}\left(\tau_{1}, \tau_{2}\right) = \frac{1}{Z_{0}^{(1)}} \sum_{n} \lambda^{n} \left\langle n \left| e^{-\beta \overline{H}} T_{\tau} \varphi\left(\tau_{1}\right) \varphi\left(\tau_{2}\right) \right| n \right\rangle = \frac{1}{\beta} \sum_{n} e^{-\varepsilon_{n} (\tau_{1} - \tau_{2})} D_{2}\left(\varepsilon_{n}\right) , \quad (D.19)$$

$$D_2(\varepsilon_n) = -\frac{\alpha}{2} \frac{1 + \alpha \lambda (-)^n}{(i\varepsilon_n)^2 - \omega^2}$$
 (D.20)

B) Fermi case:

$$\overline{H} = \alpha \omega a^{+} a, \left\{ a, a^{+} \right\} = \alpha, \ \alpha^{2} = 1, \tag{D.21}$$

$$T_{\tau} \exp \left\{ \int_{0}^{\beta} d\tau \, \left(\eta^{+} \left(\tau \right) a \left(\tau \right) + a^{+} \left(\tau \right) \eta \left(\tau \right) \right) \right\} = e^{\eta^{+} \Delta \eta} : \exp \left(\eta_{1}^{+} a + a^{+} \eta_{1} \right) :, \tag{D.22}$$

$$\eta_1^+ = \int_0^\beta d\tau \, e^{-\omega \tau} \eta^+(\tau) \,, \, \eta_1 = \int_0^\beta d\tau \, e^{\omega \tau} \eta(\tau) \,.$$
(D.23)

Calculating the generating functional, one should take into account the anticommutation properties of a and η :

$$a^2 = a^{+2} = \eta_1^2 = \eta_1^{+2} = 0.$$
 (D.24)

We have

$$Z_0^{(2)} = \sum_n \lambda^n \left\langle n \left| e^{-\beta \overline{H}} \right| n \right\rangle = 1 + \alpha \lambda e^{-\beta \omega}, \qquad (D.25)$$

$$Z^{(2)}(\eta) = \sum_{n} \lambda^{n} \left\langle n \left| e^{-\beta \overline{H}} T_{\tau} \exp \left(\int_{0}^{\beta} d\tau \left(\eta^{+} a + a^{+} \eta \right) \right) \right| n \right\rangle =$$

$$= e^{\eta^{+} \Delta \eta} \left[1 + \alpha \lambda e^{-\beta \omega} - \lambda \eta_{1}^{+} \eta_{1} e^{-\omega \beta} \left\langle 1 | a^{+} a | 1 \right\rangle \right] = Z_{0}^{(2)} \exp \left(\eta^{+} D_{3} \eta \right)$$

$$D_3(\tau_1, \tau_2) = \alpha \theta (\tau_1 - \tau_2) e^{-\omega(\tau_1 - \tau_2)} - \frac{\lambda e^{-\omega \beta - \omega(\tau_1 - \tau_2)}}{1 + \alpha \lambda e^{-\omega \beta}} =$$
(D.26)

$$= \frac{1}{\beta} \sum_{n} e^{-i\varepsilon_n(\tau_1 - \tau_2)} D_3(\varepsilon_n) , \qquad (D.27)$$

$$D_3(\varepsilon_n) = -\frac{\alpha}{2} \frac{1 - \alpha \lambda (-)^n}{i\varepsilon_n - \omega}.$$
 (D.28)

We can see, once again, that the choice $\lambda = \alpha$ implies that the Green function, just as it should be in the case of a Fermi Green function, contains only odd frequencies. However, in the case $\lambda = -\alpha$ a Fermi Green function contains only *even* frequencies. This definition leads to a Bose distribution function of a Fermion:

$$N = \frac{1}{Z_{\alpha}^{(2)}} \sum_{n} \lambda^{n} \left\langle n \left| e^{-\beta \overline{H}} \alpha a^{+} a \right| n \right\rangle = \left(e^{\omega \beta} - 1 \right)^{-1} . \tag{D.29}$$

Wick's theorem, once again, is valid for an arbitrary λ . A generalization of the above reasoning to the case of a system of particles in a space of arbitrary dimension is evident.

We finally compute the partition function and Green function of a fictitious particle, whose field operator is given by the formulas [see, (A.25), (A.26) and (A.27)]

$$C^{a}(x) = S_{ab}V_{b}(x), C^{+a}(x) = V_{b}^{+}(x)S_{ba},$$
(D.30)

$$V_{a}(x) = \int \frac{d^{3}p}{(2\pi)^{3/2} \sqrt{2|\varkappa^{00}| \nu_{p}^{a}}} \left[e^{-\omega_{p}^{a}\tau - i\vec{p}\vec{x}} d_{a}(p) + e^{\Omega_{p}^{a}\tau + i\vec{p}\vec{x}} b_{a}^{+}(p) \right], \tag{D.31}$$

$$V_a^+(x) = \int \frac{d^3p}{(2\pi)^{3/2} \sqrt{2|\varkappa^{00}|\nu_p^a}} \left[e^{\omega_p^a \tau + i\vec{p}\vec{x}} d_a^+(p) + e^{-\Omega_p^a \tau - i\vec{p}\vec{x}} b_a(p) \right], \tag{D.32}$$

$$\nu_p^a = \left[\frac{1}{\varkappa^{00}} \left(\frac{\varkappa^{0i} \varkappa^{0k}}{\varkappa^{00}} - \varkappa^{ik} \right) p_i p_k + \frac{1}{\varkappa^{00}} \mu_a \right]^{1/2} . \tag{D.33}$$

We restrict ourselves to the following values of λ_d and λ_b :

$$\lambda_b = -\lambda_d = \lambda \frac{\varkappa^{00}}{|\varkappa^{00}|}, \ \lambda^2 = 1. \tag{D.34}$$

We find

$$Z_0 = \prod_{a,p} \left(1 + \lambda e^{-\omega_p^a \beta} \right) \left(1 + \lambda e^{-\Omega_p^a \beta} \right) , \qquad (D.35)$$

$$Z(\theta) = \sum_{n} \lambda_d^{n_d} \lambda_b^{n_b} \left\langle n \left| e^{-\beta H} T_\tau \exp\left[\int_0^\beta d\tau \left(\theta^{+a} C^a(x) + C^{+a}(y) \theta^a \right) \right] \right| n \right\rangle =$$
 (D.36)

$$= Z_0 \exp\left(\theta^{+a} D_{ab} \theta^b\right) \,.$$

$$D_{ab}\left(x-y\right) = \frac{1}{Z_0} \sum_{n} \lambda_d^{n_d} \lambda_b^{n_b} \left\langle n \left| e^{-\beta H} T_\tau C^a\left(x\right) C^{+b}\left(y\right) \right| n \right\rangle =$$

$$= S_{ad} \frac{1}{\beta} \sum_{n} \int \frac{d^{3}p}{(2\pi)^{3}} e^{-i\varepsilon_{n}(\tau_{x} - \tau_{y}) - i\vec{p}(\vec{x} - \vec{y})} D_{d}(\varepsilon_{n}, \vec{p}) S_{db}^{-1}, \qquad (D.37)$$

$$D_a\left(\varepsilon_n, \vec{p}\right) = \frac{1 - \left(-\right)^n \lambda}{2\left(\varkappa^{\mu\nu} p_{\mu} p_{\nu} - \mu_a\right)}.$$
(D.38)

In (D.38), one needs to make a replacement: $p_0 \to i\varepsilon_n$. We can see that the choice $\lambda = +1$ leads to a Green function (D.38) that contains only odd frequencies. However, the case $\lambda = -1$, corresponding to the choice of λ_d and λ_b indicated in Section 5 [see, (5.31), (5.32)], leads to a Green function of Fermionic fictitious particles that contains only even frequencies.

Using the reasonings that have been presented in this appendix, one can easily see that the generating functional for an arbitrary field linear in the creation and annihilation operators can be computed with the help of Wick's theorem:

$$Z(J) = \sum_{n} \prod_{i} \lambda_{i}^{n_{i}} \left\langle n \left| e^{-\beta \overline{H}} T_{\tau} e^{J\omega} \right| n \right\rangle = Z_{0} e^{\frac{1}{2}JDJ}, \qquad (D.39)$$

where D is the Green function of the field ω :

$$D = \frac{1}{Z_0} \sum_{n} \prod_{i} \lambda_i^{n_i} \left\langle n \left| e^{-\beta \overline{H}} T_\tau \omega \omega \right| n \right\rangle. \tag{D.40}$$

This makes it possible to prove (5.17) and (5.20). In order to prove (5.22), one has to use the equations of motion and canonical commutation relations for ω .

E Appendix

Let us now perform a canonical quantization of the system of free fields A^a_{μ} , being Goldstone Bosons. In view of a tedious character of the resulting formulas, we restrict ourselves to the case $\varkappa^a_i \sim \xi^a_i$, $\alpha_{ab} = \alpha \delta_{ab}$, $\varkappa^{\mu\nu} = g^{\mu\nu}$. Then the system which consists of a multiplet of vector and scalar fields is equivalent to a set

of Abelian vector fields, each interacting only with its scalar field. The Lagrangian of this system of fields $\Omega_{\beta} = (A_{\mu}, \sigma)$ has the form

$$L = -\frac{1}{4}F_{\mu\nu}^2 + \frac{M^2}{2}A_{\mu}^2 + M\sigma\partial_{\mu}A_{\mu} + \frac{1}{2}\partial_{\mu}\sigma\partial_{\mu}\sigma + \frac{\alpha}{2}\left(\partial_{\mu}A_{\mu} + \beta\sigma\right)^2. \tag{E.1}$$

Let us present the field A_{μ} as follows:

$$A_{\mu} = V_{\mu} + \frac{1}{2} \partial_{\mu} \varphi \,, \, \partial_{\mu} V_{\mu} = 0 \,. \tag{E.2}$$

Substituting expansion (E.2) into the equations of motion for A_{μ} and σ , that follow from (E.1), we find

$$\left(\Box + M^2\right) V_{\mu} = 0, \tag{E.3}$$

$$\left(\Box - \frac{M^2}{\alpha}\right)\varphi + \left(\beta M + \frac{M^2}{\alpha}\right)\sigma = \left(1 + \frac{\alpha\beta}{M}\right)\Box\varphi - \left(\Box - \alpha\beta^2\right)\sigma = 0.$$
 (E.4)

From (E.4) it follows that

$$(\Box + \beta M)^{2} \varphi = (\Box + \beta M)^{2} \sigma = 0.$$
 (E.5)

Consequently, we can see that V_{μ} is a usual massive vector particle, whereas the fields φ and σ have to be decomposed into summands whose Fourier transformations are proportional to $\delta\left(k^2 - \beta M\right)$ and $\delta'\left(k^2 - \beta M\right)$. Substituting this decomposition into (E.4), we can see that amongst the 8 amplitudes there are only 4 arbitrary ones. In addition, the fields φ and σ can be presented in the form

$$\varphi = \int \frac{d^4x\sqrt{2k_0}}{(2\pi)^{3/2}}\theta\left(k_0\right) \left\{ e^{-ikx} \left[\delta\left(k^2 - \beta M\right)c_k + \left(\beta M + \frac{M^2}{\alpha}\right)\delta'\left(k^2 - \beta M\right)\left(c_k - d_k\right) \right] + \text{h.c.} \right\},$$
 (E.6)

$$\sigma = \int \frac{d^4x\sqrt{2k_0}}{(2\pi)^{3/2}}\theta\left(k_0\right) \left\{ e^{-ikx} \left[\delta\left(k^2 - \beta M\right) d_k + \left(\beta M + \frac{M^2}{\alpha}\right) \delta'\left(k^2 - \beta M\right) (c_k - d_k) \right] + \text{h.c.} \right\}.$$
 (E.7)

In quantum field theory, c_k , d_k and c_k^+ , d_k^+ are operators. Their commutators can be found due to the commutation relations between the fields and conjugate momenta. Let us present a qualitative analysis of finding these relations. The fact that σ and $\Pi_{\sigma} = \dot{\sigma}$ must commute with A_{μ} implies that σ and σ commute with σ , whence it follows that $[c, d^+] = 0$. In order that the commutator function should not contain σ , it is necessary that the equality $[c, c^+] + [d, d^+] = 0$ must take place. Finally, the canonical commutation relations of σ and σ yield σ is a commutator function of σ and σ yield σ in σ in σ in σ in σ in σ and σ yield σ in σ

Consequently, we find that the field Ω_{β} is presented in the form (E.2), (E.6), (E.7), where

$$V_{\mu} = \int \frac{d^4x \sqrt{2k_0}}{(2\pi)^{3/2}} \theta(k_0) \left\{ e^{-ikx} \delta(k^2 - M^2) u_k^l a_k^l + \text{h.c.} \right\},$$
 (E.8)

 \boldsymbol{u}_k^l being three orthonormalized transversal polarization vectors. The creation and annihilation operators obey the commutation relations

$$\left[a_k^l, a_p^{+l'}\right] = \delta_{ll'} \delta\left(\vec{k} - \vec{p}\right), \quad \left[d_k, d_p^+\right] = -\left[c_k, c_p^+\right] = \delta\left(\vec{k} - \vec{p}\right), \tag{E.9}$$

the remaining commutators being equal to zero. By direct calculation, we can prove that all the canonical commutators between the fields Ω_{β} and the canonical momenta constructed from Lagrangian (E.1) are fulfilled.

The Hamiltonian of the system equals to

$$H = \int d^{3}k \left[\sum_{l} \omega_{k} a_{k}^{+l} a_{k}^{l} + \Omega_{k} d_{k}^{+l} d_{k}^{l} - \Omega_{k} c_{k}^{+} c_{k} \right] + \frac{\beta M + \frac{M^{2}}{\alpha}}{2\Omega_{k}} \left(c_{k}^{+} - d_{k}^{+} \right) \left(c_{k} - d_{k} \right) ,$$

$$(E.10)$$

$$\omega_{k} = \sqrt{\vec{k}^{2} + M^{2}} , \ \Omega_{k} = \sqrt{\vec{k}^{2} + \beta M} ,$$

$$(E.11)$$

The δ' -function can be presented as $\frac{1}{k_0}\partial/\partial k_0\delta\left(k^2-\beta M\right)$ and then integrated by parts. As a result, we find that the fields φ and σ can be presented in the form (E.6), (E.7), with the following replacement:

$$\delta'\left(k^2 - \beta M\right) \to \left(\frac{1}{4k_0^2} + \frac{it}{2k_0}\right) \delta\left(k^2 - \beta M\right) .$$
 (E.12)

This representation is useful in the case $\beta = 0$, when the δ' -function needs an additional determination. In the general case, described by the Lagrangian

$$L = -\frac{1}{4}F^{a}_{\mu\nu}F^{a,\mu\nu} + \frac{M_{0}^{2}}{2}A^{a}_{\mu}A^{a,\mu} + M^{2}\sigma^{a}\partial_{\mu}A^{a,\mu} + \frac{1}{2}\partial_{\mu}\sigma^{a}\partial^{\mu}\sigma^{a} + \frac{1}{2}t^{a}\alpha_{ab}t^{b},$$

$$t^{a} = \varkappa^{\mu\nu}\partial_{\mu}A^{a}_{\nu} + \varkappa_{ab}\sigma^{b},$$
(E.13)

one should proceed in a similar way. Let us now introduce a field V_{μ}^{a} ,

$$V_{\mu}^{a} = A_{\mu}^{a} - \frac{1}{M_{a}} \partial_{\mu} \sigma^{a} , \ t^{a} = \varkappa^{\mu\nu} \partial_{\mu} V_{\nu}^{a} + \frac{1}{M_{a}} \left(\varkappa^{\mu\nu} \partial_{\mu} \partial_{\nu} \delta_{ab} + M_{a} \varkappa_{ab} \right) \sigma^{b} . \tag{E.14}$$

In terms of this field, the equations of motion acquire the form

$$\Lambda_{\mu\nu}^{ab} V^{b,\nu} = 0, \quad M^a \partial_{\mu} V^{a,\mu} + \varkappa_{ba} \alpha_{bc} t^c = 0,
\Lambda_{\mu\nu}^{ab} = \delta_{ab} \left[g_{\mu\nu} \left(\Box + M_a^2 \right) - \partial_{\mu} \partial_{\nu} \right] + \widehat{\partial}_{\mu} \partial_{\nu} \varkappa_{ab}^{-1}, \quad \widehat{\partial}_{\mu} = \varkappa_{\mu}^{\nu} \partial_{\nu}.$$
(E.15)

We now have to find the spectrum of the system. For this purpose, we need to calculate det Λ , namely,

$$\ln \det \Lambda = \operatorname{Sp} \ln \Lambda = \operatorname{Sp} \ln \lambda^{-1} (1 + \lambda \Lambda) \Big|_{\lambda = \infty} =$$

$$= \operatorname{Sp} \ln \lambda + \sum_{k} \operatorname{Sp} (-)^{k} \lambda^{k} \left[\left(g_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\Box} \right) \left(\Box + M_{a}^{2} \right) \delta_{ab} + \left(\frac{\partial_{\mu}\partial_{\nu}}{\Box} M_{a}^{2} \delta_{ab} + \widehat{\partial}_{\mu}\partial_{\nu} M_{a} \varkappa_{ab}^{-1} \right) \right]^{k} = \operatorname{Sp} \ln \lambda + \frac{3}{4} \operatorname{Sp} \delta_{\mu\nu} \delta_{ab} \ln \left[1 + \lambda \left(\Box + M_{a}^{2} \right) \right] +$$

$$+ \frac{1}{4} \operatorname{Sp} \delta_{\mu\nu} \ln \left[1 + \lambda M \varkappa^{-1} \left(\widehat{\partial}_{\sigma} \partial^{\sigma} + \varkappa M \right) \right] \Big|_{\lambda = \infty} =$$

$$= \det^{-1} \varkappa_{ab} \det \left(\widehat{\partial}_{\mu} \partial^{\mu} + \varkappa_{ab} M_{b} \right) \prod_{a} M_{a} \det^{3} \left(\Box + M_{a}^{2} \right) . \tag{E.16}$$

Thus, the spectrum of the system is determined by solutions of the equations

$$\Box + M_a^2 = 0 \,, \, \widehat{\partial}_\mu \partial^\mu + \varkappa_{ab} \mu_b = 0 \,, \tag{E.17}$$

where μ_a stand for the eigenvalues of the matrix $m_{ab} = M_a \varkappa_{ab}$. Taking account of the expression (E.14) for t^a , we can see that the field V^a_μ has to be decomposed in $\delta \left(k^2 - M_a^2\right)$ and $\delta \left(\varkappa^{\mu\nu}k_\mu k_\nu - \mu_a\right)$, whereas the field σ should be decomposed in $\delta \left(k^2 - M_a^2\right)$, $\delta \left(\varkappa^{\mu\nu}k_\mu k_\nu - \mu_a\right)$ and $\delta' \left(\varkappa^{\mu\nu}k_\mu k_\nu - \mu_a\right)$.

Let us also present the expressions for (Wick's) field propagators. The simplest way to find them is to calculate the Gaussian functional integral of L. In addition, we can see that if one introduces sources for the fields V^a_{μ} and t^a (rather than those for A^a_{μ} and σ^a) the integrals over V^a_{μ} and t^a factor out and can be easily computed. As a result, we have

$$\left\langle V_{\mu}^{a} V_{\nu}^{a} \right\rangle = i \delta_{ab} \left(g_{\mu\nu} + \frac{\partial_{\mu} \partial_{\nu}}{M_{a}^{2}} \right) \frac{1}{\Box + M_{a}^{2}}, \tag{E.18}$$

$$\langle V_a^a t^a \rangle = 0, \tag{E.19}$$

$$\langle t^a(x) t^a(y) \rangle = i\alpha^{ab}\delta(x - y) . \tag{E.20}$$

Hence, the propagators of the fields A^a_μ and σ^a are found with the help of the relations

$$\sigma^{a} = (\varkappa^{\mu\nu}\partial_{\mu}\partial_{\nu}\delta_{ab} + m_{ab})^{-1} M_{b} \left(t^{b} - \varkappa^{\mu\nu}\partial_{\mu}V_{\nu}^{b}\right) ,$$

$$A_{\mu}^{a} = V_{\mu}^{a} + \frac{1}{M_{a}}\partial_{\mu} \left(\varkappa^{\mu\nu}\partial_{\mu}\partial_{\nu}\delta_{ab} + m_{ab}\right)^{-1} M_{b} \left(t^{b} - \varkappa^{\mu\nu}\partial_{\mu}V_{\nu}^{b}\right) . \tag{E.21}$$

F Appendix

In this appendix, we deduce, for the sake of completeness, the Ward identities in case the Green functions include the fictitious fields. Deriving these identities is quite similar to the corresponding calculation of Section 3, and, therefore, we are not going to present a detailed analysis.

Consider the generating functionals

$$Z_{C_{\eta}^{d}C_{x}^{+a}} = \left\langle 0 \left| T \exp\left(i\hat{Q}\right) c^{d}\left(y\right) c^{+a}\left(y\right) \right| 0 \right\rangle, \tag{F.1}$$

$$Z = \left\langle 0 \left| T \exp\left(i\hat{Q}\right) \right| 0 \right\rangle, \tag{F.2}$$

where

$$\hat{Q} = Q + \theta^{+a}C^a + C^{+a}\theta^a. \tag{F.3}$$

An analogue of formula (3.12) now takes the form

$$\int dy \, t^{f}(y) \, \alpha_{fc} T^{cd} Z_{CC^{+}} = \int dy \left\langle \left[g \theta^{+b}(y) \, f^{bdn} C^{n}(y) - Q_{R}^{d}(y) \right] C^{d}(y) \, C^{+a}(y) \right\rangle +$$

$$+ \int dy \left\langle \left[-g \varkappa^{\mu\nu} \nabla_{\mu}^{db'} \left(\partial_{\nu} C^{+b}(y) \, f^{bb'f} C^{f}(y) \right) + g^{2} C^{+b}(y) \, \varkappa^{b} \Gamma^{f} \Gamma^{d} \varphi(y) \, C^{f}(y) \right] C^{d}(y) \, C^{+a}(x) \right\rangle. \quad (F.4)$$

In this appendix, we use the following notation:

$$\langle \ldots \rangle = \left\langle T(\ldots) \exp\left(i\hat{Q}\right)\right\rangle.$$
 (F.5)

As in Section 3, we need to carry out an integration by parts in the second term of (F.4). We, however, must take into account that \hat{Q} depends on C and C^+ , and, therefore, ∂_0 does not commute with the symbol of T-product. We need to extract ∂_0 from the symbol of T-product, then integrate by parts, using the antisymmetry of the operators C (as has been done in Appendix B; one only has to remember that ∂_0 now stands aside from the symbol of T-product), then integrate by parts once again, and finally insert ∂_0 in the symbol of T-product. In transposing ∂_0 with the symbol of T-product, we need formula (3.3), while also making the replacement $Q \to \hat{Q}$. As a result, the Ward identities have a surprisingly simple form:

$$i\alpha_{ab}t^{b}(x)Z = -\left\langle \int dy \, Q_{R}^{b}(y) \, C^{b}(y) \, C^{+a}(x) \right\rangle + \left\langle \int dy \, t^{b}(y) \, \alpha_{bd}\theta^{d}(y) \, C^{+a}(x) \right\rangle -$$

$$-\frac{g}{2} \left\langle \int dy \, \theta^{+b}(y) \, f^{bdf}C^{d}(y) \, C^{f}(y) \, C^{+a}(x) \right\rangle. \tag{F.6}$$

This identity has been verified by an explicit calculation in quantum electrodynamics (within a gauge in which particles are free) as well as in the first perturbative order of a non-Abelian theory.

It should be noted that the process of deriving the Ward identities with the help of the usual procedure of a non-local change of variables in the functional integral [11] for the generating functional leads to a much more involved expression than (F.6). It is interesting, however, that there exists another change of variables (*supertransformation*), that allows one to obtain (F.6), as well as the usual Ward identity.

Consider the generating functional

$$Z = \int dA \, d\varphi \, d\psi \, d\overline{\psi} \, dC \, dC^{+} e^{iL + i\widehat{Q}} \,, \tag{F.7}$$

where the expression for L is given by formula (2.1). Let us subject (F.7) to the change of variables (2.13)–(2.14), and let us choose the gauge parameter as follows:

$$\Lambda^{a}(x) = \mu C^{a}(x) , \qquad (F.8)$$

where μ is an anticommuting object. Notice that the choice (F.8) leads to an exact form of transformations (2.13)–(2.14), since $\mu^2 = 0$. Besides, let us make a change of the field C^+ , namely,

$$C^{+a} \to C^{+a} - \mu \alpha_{ab} t^b(x) . \tag{F.9}$$

The corresponding variation of the Lagrangian reads

$$\delta L = \int dx C^{+a} \delta T^{ab} C^b = -\frac{\mu g}{2} \int dx C^{+a} T^{ab} f^{bdf} C^d C^f , \qquad (F.10)$$

where δT^{ab} is the result of taking a variation of T^{ab} , whereas a transition from the first equality to the second one in (F.10) is compensated by the following transformation of the field C:

$$C^{a}\left(x\right) \to C^{a}\left(x\right) - \frac{g}{2}f^{adb}\Lambda^{d}\left(x\right)C^{b}\left(x\right),$$
 (F.11)

where Λ is given by formula (F.8). As a result, we find that Lagrangian (2.1), or, more exactly, the action, is invariant under (super)transformations (2.13)–(2.14), (F.11) with the parameter Λ defined by formula (F.8), whereas the Jacobian of this change equals to 1. Therefore, variation affects only the term with the sources, and so we obtain the following relation:

$$\int dA \dots \int dy \left[\frac{g}{2} \theta^b f^{bdf} C^d C^f + Q_R^b C^b - t^b \alpha_{bc} \theta^c \right] e^{iL + i\widehat{Q}} = 0.$$
 (F.12)

Differentiating (F.12) by $\delta/\delta\theta^a(x)$, we obtain the Ward identity in the form (F.6).

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